INTEGRAL EXTREME POINTS

BY

ARTHUR F. VEINOTT, JR. and GEORGE B. DANTZIG

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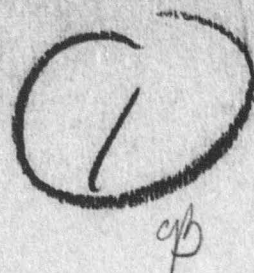
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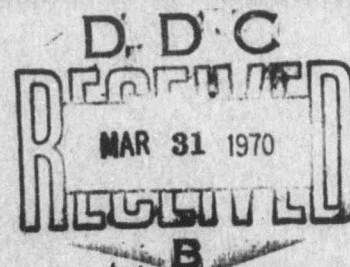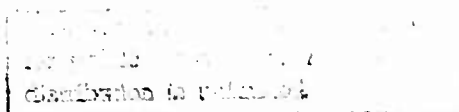
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Let A be a given integral matrix and let $X(A,b) = \{x:Ax = b, x \geq 0\}$ and $X^*(A,b) = \{x:Ax \leq b, x \geq 0\}$. If A has r rows, all linearly independent, a subset of the columns of A is called a basis if its rank is r. In this event, an obvious sufficient condition for the extreme points of $X(A,b)$ to be integral for all integral b is that the determinant of each basis equals $+1$ or -1. The purpose of this note is to give a short proof that this condition is necessary and to obtain thereby a substantially simpler proof of an important result of Hoffman and Kruskal (1956, p. 225). Their result is that if A is an integral matrix, then the extreme points of $X^*(A,b)$ are integral for all integral b if and only if A is unimodular (i.e., each minor of A equals 0, $+1$, or -1).

<u>Theorem</u>.

If A is an integral matrix having linearly independent rows, the following are equivalent.

1° The determinant of every basis equals +1 or -1.

2° The extreme points of $X(A,b)$ are integral for all integral b.

3° Every basis has an integral inverse.

Proof.

$1^o \Rightarrow 2^o$. (We repeat the standard proof of this for completeness.) Suppose b is integral. Let x be an extreme point of $X(A,b)$, B an associated basis, and x_B the corresponding components of x (the remaining components of x vanish). Since $Bx_B = b$ and det B $= \pm 1$, by Cramer's rule x_B is integral.

$2^o \Rightarrow 3^o$. Let B be a basis. Let y be any integral vector for which $z \equiv y + B^{-1}1_i \geq 0$ where 1_i denotes the ith unit column vector. Then $Bz = By + 1_i \equiv b$ is integral and z contains the nonvanishing components of an extreme point of $X(A,b)$ so z is integral by hypothesis. Thus $z-y = B^{-1}1_i$, the ith column of B^{-1}, is integral. Since this is so for all i, B^{-1} is integral.

$3^o \Rightarrow 1^o$. Let B be a basis. By hypothesis B and B^{-1} are integral, so det B and det B^{-1} are nonvanishing integers such that (det B)(det B^{-1}) $= 1$. Hence det B $=$ det $B^{-1} = \pm 1$.

Corollary. (Hoffman and Kruskal)

If A is an integral matrix, the following are equivalent.

1* A is unimodular.

2* The extreme points of $X^*(A,b)$ are integral for all integral b.

2

3* Every nonsingular submatrix of A has an integral inverse.

Proof.

Let $A' = (A,I)$ have r rows; these are linearly independent. Upon replacing A by A' in the theorem, one sees that the statements 1°, 2°, 3° about A' are equivalent to the corresponding assertions in the corollary about A. For example, $1*$ follows readily from 1° for if C is any nonsingular submatrix of A of rank $r-k$, then a basis B in A' can be found, after permuting rows, of the form

$$B = \begin{pmatrix} C, & 0 \\ D, & I_k \end{pmatrix}$$

where I_k is a $k \times k$ identity matrix. Then $\det B = \det C$, so that $\det B = \pm 1$ if and only if $\det C = \pm 1$.

Remark. We can obtain other corollaries by noting that if any one of the matrices A, A^T, $-A$, (A,A), or (A,I) is unimodular, then so are all the others. To illustrate, consider the set $X*(M,b)$ as defined earlier with

$$M = \begin{pmatrix} A \\ -A \\ I \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} \bar{b} \\ -\underline{b} \\ c \end{pmatrix}$$

where A and b are integral. This set is identical with the set $X**(A,b)$ defined by

$$X**(A,b) = \{x : \underline{b} \le Ax \le \bar{b}, \ 0 \le x \le c\}.$$

3

Notice that M is unimodular if and only if A is unimodular. Thus we may replace $X^*(A,b)$ in 2^* by $X^{**}(A,b)$ to obtain another result given in Hoffman and Kruskal (1956, p. 225).

Reference

HOFFMAN, A.J., and J.B. KRUSKAL, (1956), "Integral Boundary Points of Convex Polyhedra", Chapter 13, in H.W. Kuhn and A.W. Tucker (eds.), Linear Inequalities and Related Systems, Princeton University Press, Princeton, N.J.

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13. ABSTRACT

It is shown that if A is an integral matrix having linearly independent rows, then the extreme points of the set of nonnegative solutions to $Ax = b$ are integral for all integral b if and only if the determinant of every basis matrix is ± 1. This provides a short proof of the Hoffman-Kruskal theorem characterizing unimodular matrices, i.e., matrices in which the determinant of each nonsingular submatrix is ± 1. Their theorem is that if A is integral, then A is unimodular if and only if the extreme points of the set of nonnegative solutions to $Ax \le b$ are integral for all integral b .

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